Students' solution strategies in spatial rotation tasks

Strategie di soluzione degli studenti in attività di rotazione spaziale

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Abstract. *This study focuses on the solution strategies of 6 Greek secondary students, one from each different grade of secondary education, in 5 mental rotation tasks from which emerges as a usual strategy, a visual-analytical approach within the harmonic type of reasoning, where the visual component came first and the analytical followed. In two-dimensional tasks, the students mainly followed a visual strategy, which is often accompanied by the use of gestures and body movements, in contrast to the three-dimensional tasks, where the need appears for visualization to be combined with a verbal description of transformations of the shape, in a mental rotation. At the same time, through problem solving, both a correlation of human memory with visualization and the development of creative approaches by students are highlighted; one of them was not able to distinguish that the front and back view of an object are not the same but reversed. Thus, through all the approaches, the need for students' action either for interpretation or for construction becomes clear.*

Keywords: mental rotation, harmonic type of reasoning, spatial processing ability, required action.

Sunto. *La presente ricerca si focalizza sulle strategie risolutive di sei studenti greci di scuola secondaria, uno per ciascun livello del percorso d'istruzione secondaria, in riferimento a cinque compiti che richiedono di immaginare la rotazione di un oggetto e dai quali emerge come strategia usuale un approccio visuale-analitico nell'ambito del ragionamento armonico, in cui la componente visuale precede quella analitica. Nei compiti bidimensionali gli studenti hanno seguito principalmente una strategia di visualizzazione, che spesso è stata accompagnata dall'uso di gesti e di movimenti del corpo, a differenza di quanto avvenuto nei compiti tridimensionali, dove sembra manifestarsi la necessità che la visualizzazione sia accompagnata da una descrizione orale di trasformazioni della forma, risultanti da una rotazione immaginaria. Nello stesso tempo, attraverso la risoluzione dei problemi, viene messa in evidenza una correlazione tra la memoria umana e la visualizzazione e lo sviluppo di approcci creativi da parte degli studenti; uno di loro non è stato in grado di riconoscere che la vista dal di fronte e di dietro non sono uguali, ma rovesciate l'una rispetto all'altra.*

Così, trasversalmente a tutti gli approcci, diventa chiara la necessità della messa in atto da parte degli studenti di azioni finalizzate all'interpretazione o alla costruzione.

Parole chiave: rotazione mentale, ragionamento di tipo armonico, capacità di processamento spaziale, azione richiesta.

Resumen. *Este estudio se centra en las estrategias de solución dadas por cada uno de los 6 estudiantes de secundaria griegos (uno por cada grado de escolaridad) que intervinieron en la prueba, en 5 tareas de rotación mental de las cuales surgen, como estrategia habitual, un enfoque visual-analítico dentro del tipo armónico de razonamiento, en el cual la componente visual llega en primer lugar seguida de la componente visual. En las tareas bidimensionales, los estudiantes siguieron principalmente una estrategia visual, que a menudo va acompañada del uso de gestos y movimientos corporales, en contraste con las tareas tridimensionales, donde aparece la necesidad de que la visualización se combine con una descripción verbal de las transformaciones de la forma, en una rotación mental. Al mismo tiempo, a través de la resolución de problemas, se destaca tanto una correlación de la memoria humana con la visualización como el desarrollo de enfoques creativos por parte de los estudiantes; uno de ellos no fue capaz de distinguir que las vistas frontal y trasera de un objeto no son las mismas, pero invertidas. Por lo tanto, a través de todos los enfoques, la necesidad de la acción de los estudiantes, ya sea para la interpretación o para la construcción, se hace evidente.*

Palabras claves: rotación mental, tipo armónico de razonamiento, habilidad de procesamiento espacial, acción requerida.

1. Introduction

Applying transformations and using visualization and spatial reasoning are two of the principal standards for geometry in the standards for school mathematics (National Council of Teachers of Mathematics, 2000). Spatial ability is defined by Lohman (1996) as the ability of the individual to produce, maintain, recall, and transform well-structured visual images. Moreover, it is defined by McGee (1979) as the ability to form mental images and manipulate them in the individual's thinking. In addition to them, Guay and McDaniel (1977) defined as low-level spatial ability, the ability to visualize – but not mentally transform – two-dimensional objects, and as high-level spatial ability, the ability to visualize three-dimensional shapes, and to mentally manipulate them. In addition, Gorgorió (1998) proposes the use of the construct "spatial processing ability" instead of the construct "visual processing ability", to denote not only the individual's ability to imagine spatial objects, their relationships, and transformations, but also their ability to decode these data visually as well as the ability to encode them in verbal or mixed ways. Linn and Petersen (1985) identified three factors that define the ability to perceive spatial concepts. One of them was "mental rotation" which was defined as the ability to quickly mentally rotate two-dimensional and three-dimensional shapes. In addition, mental rotation involves the mental processing of an image or object by observing how it will look from a different point of view when rotated rather than reflected (Michaelides, 2006). This study seeks to investigate what are the characteristics of students' solving strategies when engaging in mental rotation tasks, to analyze the functionality and effectiveness of these strategies as a function of task characteristics and to note any difficulties that arise.

2. Theoretical perspectives

Much of the research in the field of spatial ability is quantitative and based on factor analysis, on equation structural modelling and other statistical methods (Gagatsis & Kalogirou, 2013; Kalogirou, Elia, & Gagatsis, 2009; Kalogirou & Gagatsis, 2011, 2012). This research provides classifications that do not support the hypothesis that all individuals solve all tasks in a test in the same way (Lohman, 1979). Thurstone (1938) notes that there is evidence that individuals use different strategies to solve spatial problems. This view has led to many studies, where individuals have been asked to indicate what strategies they used to solve such problems (Lohman, 1979; Lohman & Kyllonen, 1983). In a study of gifted students, Kruktetskii (1976) proposed a classification of student solving strategies and distinguished three types of students, based on how they interpret the world mathematically and their preferences for problem solving. The research showed that a small percentage of students followed an analytical-productive and verbal-logical procedure (analytic type) or a visual-pictorial procedure (geometric type). In contrast, the majority of students used both abstract and pictorial representations (harmonic type) depending on the context of the problem (Michaelides, 2006). Thus, the conclusion that mental images and verbal processes not only do not function independently but interact as they have supportive functions (Fischbein, 1993; Lean & Clements, 1981; Lohman, 1979; Paivio, 1971) and this result has led to the argument that spatial ability is not identified exclusively with the processing of virtual information. This argument was reinforced by Gorgorió (1998) through the use of visualization, in the study of strategies for solving mental rotation tasks. Gorgorió (1998) proposed a distinction between visual solution strategies, where one can deduce from the student's observations and explanation that visual images had been used as an essential part of the solution, and non-visual or analytic strategies, where such images are not used, but an argument is developed to justify the solution strategy. In a visual strategy (holistic approach), verbal features are not specific or detailed enough and are often accompanied by gestures of students and body movements in an attempt to describe a mental movement such as rotation. Moreover, other research has noted the importance and role of the body and in particular perceptual-motor activities in learning mathematics (Lakoff & Núñez, 2000; Nemirovsky et al., 1998).

Elia, Gagatsis, and van den Heuvel-Panhuizen (2014) studied the role of gestures in making connections between space and shape aspects and their verbal representations in the early years. The same researchers studied the development of understanding of shapes and space with a focus on visualization (Elia, van den Heuvel-Panhuizen, & Gagatsis, 2018).

In fact, the use of gestures often reveals strategies that are difficult to articulate (Garber, Alibali, & Goldin-Meadow, 1998; Goldin-Meadow, Alibali, & Church, 1993). On the contrary, in an analytic strategy (partial approach), their attention is focused on parts or properties of the representations such as left-right part, relative position, etc., in an attempt to formulate an argument.

Among the characteristics considered to be able to modify or influence students' strategies, Gorgorió (1998) introduced the term "required action" to describe the most important characteristic, that is, the action required by the individual, as defined by Leinhardt et al (1990), to solve the problem. Thus, according to the authors, this required action can be one of interpretation or of construction. In the first case, the student must obtain information from an object or representation and such an interpretation work is considered when the student asked to react to the view of an integrated geometric action, as in the case of matching an element with someone representing the application of a geometric transformation into it, like mental rotation, in the context of multiple-choice questions. Instead, in the second case, the student acts to create a new object, constructing or representing it. That is, given the initial element, the student must create the final, performing the required geometric action himself, either mentally or tactilely (Gorgorió, 1998). As Gorgorió (1998) points out, the relationship between the two actions is such that interpretation work does not require construction while construction is often based on some kind of interpretation.

3. Method

3.1. Participants

This study was addressed to 6 Greek secondary students, one from each grade. The age of the participants ranged from 12-17 years.

3.2. Instruments: Questionnaire and interview

The study is completed in 2 phases. Initially, students are asked to answer a 5-task questionnaire (Appendix A), which aims to investigate the solution strategies that students apply to two- and three-dimensional tasks. The first two of them concern two-dimensional objects and the other three concern three-dimensional objects, which mainly concern cubic constructions. For the first four questions, students have to answer in multiple choice questions, through a number of images that are image variations provided by standardized tests (Eliot & Smith, 1983; Michaelides, 2006). In these first four questions, students are asked to isolate a representation that is the same or different from the original shape. The last task requires a problem solving, where the student is asked to explain the strategy used, which is the most personal ability to process virtual information (Bishop, 1983). In the second stage, the students were invited to short interviews (Appendix B), where they were asked to explain what prompted them to follow these solutions. The discussions took place online, on a communication platform, were videotaped and any notes made by the students were recorded. The wording of the questions was simple so as not to influence the students. The students' answers described in each episode are accompanied by a 1×4 vector of the form (Si, Ti, S or F, V or A or VA) stating that the student Si worked on the task Ti with success (S) or failure (F) having a visual (V) or analytic (A) or visual-analytic (VA) approach. Also, the code Si corresponds to the student of the i grade of secondary education.

4. Results

During the analysis of data, it was observed that none of the participants followed a unique type of strategy – either visual or analytical – for all five tasks. The constant alternation of strategies in different tasks or even their combination in the same task was the usual strategy of students, mainly for the three-dimension tasks (Table 1).

Table 1
Students' strategy in each task

Description of task			**Strategy**		
Task	Type	Dimension	Visual	Analytic	Visual-Analytic
1	*Multiple Choice*	2	5	0	1
2	*Multiple Choice*	2	5	0	1
3	*Multiple Choice*	3	0	3	3
4	*Multiple Choice*	3	0	4	2
5	*Problem solving*	3	0	0	6

Indeed, these tasks were more difficult for the students and in particular the fifth task (the rotated dice). For the two-dimensional tasks, a visual strategy was mainly applied which soon led all the students to correct answers. On the contrary, in the three-dimensional tasks, analytic strategies were observed to a small extent, focusing on specific characteristics of the shapes and to a greater extent, combined strategies of the two above, mainly in the problem solving but also in the third and fourth task. It is worth noting that only the second-grade student (S2) was able to answer all the questions correctly, justifying his point of view.

4.1. *Features of visual strategies*

The visual strategies followed by the students are generally characterized by a lack of verbal description of the features of the shape, while 5 of the students used a gesture either by moving their hand or even their whole body by bending, in their attempt to watch and identify the transformations of the shape with some of the given options they had.

In fact, one of them (S2) instead of being limited to the circular movement of the hand, wanted to explain his point of view by pretending with his hands, each arrow in the first image (Figure 1) justifying that one hand can never be on the other side, as in the last photo of Figure 1.

Figure 1. Use of gestures for representation of arrow movements.

In addition, another student (S1) formed a circle with both hands and wanted to rotate it (fig. 2), essentially emphasizing that a continuous check was made to identify the rotating initial shape with the given options.

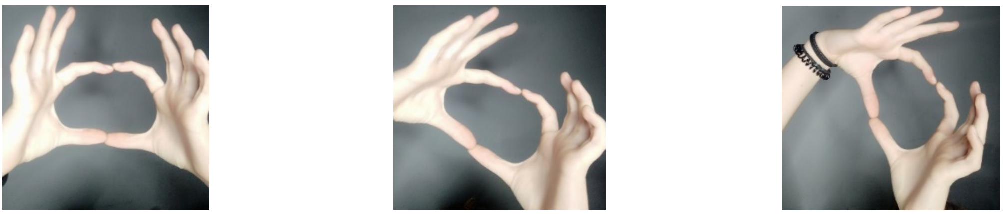

Figure 2. Use of gestures for representation of circular movement.

The excerpt from the justification for S2 is indicative:

R: Why is C?

S2: If we turn the circle, rotate it, everything will come out the same as above

except for C.
R: In fact… (I remain silent for a while) that is, it can never take position C?
S2: Look to put it more simply. Look at my two hands. The left is the black arrow and the right is the red. If I move them in a circle, I can never have C because I would have to have my hands otherwise…
(S2, T1, S, V)

Of particular interest is the justification of the correct answer to the first task by the student S3, who used her hand to first show the position of the original shape and then showed her hand in a position of reflection to the original position (Figure 3), saying that "It's as if we have reversed it and that my hand can never come to this position as it was originally".

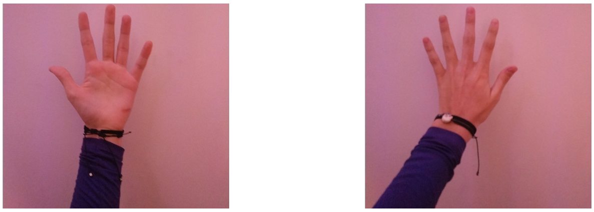

Figure 3. Use of gestures to show reflection

In addition, during the solving of the second task, the older student S6 used the phrase "axis of rotation" while giving an interesting parallel with the design of a circle without geometric instruments but with the use of a clip, showing to apply a continuous circular motion, which in a snapshot it is identified with figure B. Indicatively mentioned:

R: Why B?
S6: I saw the original, I saw its axis of rotation, that is, basically if we could press it in the middle and turn it…
R: Where in the middle?
S6: There at the end of the straight section, in the line that it has and if we turn it then it gives us the B. This is like the way I make a circle without diabetes, holding one end in one clip and putting a pencil in the other.
(S6, T2, S, V)

Another major feature of the visual strategies observed in two students is the dominance of intuition due to their observation.

R: What do you think?
S1: It is B. I saw it directly.
R: So, you just saw this or did you check the rest?
S1: Now that I see the rest… it's all upside down, so that's what I said at the beginning.
(S1, T2, S, V)

4.2. Features of analytic strategies

The analytic strategies were applied to some of the three-dimensional tasks. In these tasks the students examined specific parts and features of the shapes, expressing a sequential way of thinking, based mainly on well-formulated reasoning based on logical arguments and comparisons of parts of the shapes rather than holistic manipulation of the shape visually. These justifications were mainly expressed either through the numbering of the cubes in a row or a column or even in the whole cubic construction but also in the observation of the position of some subsections of the figure.

> S5: I counted all the cubes on the original shape and it is 18…. Then I saw the ones below and they are all 17 except B which is 18
> **(S5, T3, S, A)**

Finally, it is worth noting that one of the students had difficulty distinguishing that the front and back view of the shape are not the same but reversed.

> S4: You see from here, yes (he thinks) … that is, it can be the same… uh .. it will be the same! Yes since it does not want something, then it will be like C? Yes, the C!
> R: C?
> S4: The back view is the same as the front view, it will not change because if it changed, we would see it in the original shape. Basically, my object remains the same. I did it with my hands, I put my two hands and I see it with my thumb up and I saw that if they were opposite, they would not join so I would see something else.
> **(S4, T4, F, A)**

4.3. Features of combination of visual and analytic strategies

The combined strategy of the visual and the analytic approach at the same time was applied mainly in the three-dimensional schemes and in the problem solving but also in two solutions of two-dimensional tasks.

> R: You tell me B. Why?
> S5: If you pay attention to the right angle that is formed and try to rotate it, the shape C can never appear.
> Now if we look at D, then it is the same as C and A is also like that. So, it's B.
> **(S5, T2, S, VA)**

It is remarkable that in almost all cases where both a visual and an analytic strategy were followed at the same time, with the visual always preceding the analytic one, the students correctly solved the task, given to them. In fact, the student S6, during the execution of task 3, initially stated that all five shapes are the same, approaching the solution visually, as he confirmed in his interview and then recalled his answer focusing on individual parts of the shape, approaching the solution, but again without success, showing that he did not combine the two approaches.

S6: They are all the same!
R: For example, if I ask you about any of these, can you tell me what you did?
S6: I took the original shape and, in my mind, I tried to turn it so that it is the same as A for example and I did it in all the shapes to see which one is the same.
R: In fact, is it the same?
S6: Yes, as the rest.
R: Ok, so you are telling me that all 5 shapes are the same.
S6: Aa! Just a minute! Now I noticed this! Let's say C, from the top the second step below has one step while the one above (original) has two. So, it is a different shape and the same goes for A. So, I conclude that it is B and D.
R: Do you think that B and D are the same as the original shape?
S6: Yes, I counted the steps and they have the same number of cubes.
(S6, T3, F, A)

In addition to the above excerpt, student S1 referred exactly that firstly rotate the shape B so that it has the position as the original shape and secondly counted the cubes on top of the shape, where this part of the shape seems like a ladder. So, S1 follow firstly a visual approach that it is combined later with an analytic one.

R: It's B, you tell me. Why should this be?
S1: Well, if we return the shape (meaning B) to its normal form (meaning the position of the original shape)…
R: When you say return what do you mean?
S1: Rotate it, how do I say it?
R: Let's rotate it, in fact… and?
S1: ... I counted the cubes on top that look like a ladder and saw that most of the cubes are at the beginning and the rest A, C and D…
R: Beginning? What is the beginning?
S1: The beginning is from left to right and because it has more cubes to the left I say that it is B.
(S1, T3, S, VA)

Another remarkable approach was made by the student S3 during the solution of task 3, where after first making a mental rotation as he confirmed in his interview, he assigned a number to each column of the cubic structure and emphasized the arrangement of numbers, seeing construction as a mapping.

R: Why not someone other than B? As A, for example?
S3: I counted in the normal shape (original), the cubes in each row vertically (meaning column). The original shape goes 1 cube, 2, 2, 3, 3, 4, 3, 2 while in A if I bring it otherwise it goes 1, 2, 2, 3, 4, 3, 2.
R: And you did that for everything?
S3: Yes, I turned the shapes, brought them in a straight line (position of the original shape) and measured in each row. Only B fits.
(S3, T3, S, VA)

Of particular interest is the approach of solving the fifth task (problem) by the student S4, who seemed to confuse the front and back view of a shape earlier.

S4 creatively tried to reconstruct the die from the beginning, something that was not clear at first, but he claimed in his post-task interview, based on the two images he provided. Student S6 tried something similar, starting with the faces of the first die. In these cases, the task resolution time was extremely shorter compared to other student approaches.

> R: I would also like to ask you: Did you do something different from what you told me in the description? Did something I do not know help you?
> S4: Yes, the truth is that I tried to see how the die are.
> R: In what way?
> S4: I started with an empty cube and based on what I saw from the pictures, I started to place the numbers in the right places, it is as if I had the cube in front of me and turned it where I wanted. After I put the opposite sides 1-6 and 3-4 I was left with 2-5, so I had the die as an image and as soon as I turned it, I first raised the 4 and then the 1 on the right, so I got the 5 ahead. Like I have a Rubik's cube.
> **(S4, T5, S, VA)**

In a similar way of solution, student S6 suggested a reconstruction of die with the exception that the view of first die helps you to begin the process of placement of the suitable indications of it. In addition to this, S6 confirms at the end of his point that he founded all the faces of first die.

> S6: I first saw the two dice that have both rolled 2. The first one has 1 and 3 while the other one has 6 and 4, so we have actually rolled 2 in a different direction, so if we turn it clockwise twice in order for 3 and 1 to leave, then based on the positions of 6 and 4, 6 must be opposite 1 and 4 opposite 3, so I have really found all the positions (faces) on the first die.
> R: Yes, yes, go on…
> S6: ...so now, tilt it to the right (axis of rotation: pair opposite sides 1 and 6), it will take out 4, but in front it will be 1 and 2, so to go where 1 is in the last shape, you have to turn it clockwise (axis of rotation: pair opposite sides 4 and 3) and the bottom which is the number 5 to the left of 1 will go where the side we are looking for is.
> **(S6, T5, S, VA)**

It is worth noting that problem 5 initially caused confusion about finding the requested face, revealing an inability of students to memorize both faces and die transformations. Over time though logical arguments were made about the opposite faces and the perception of the constant presence of 2 on the upper face of the two-given dice. However, three of the students approached the solution stating with certainty that the requested face is 2 or 5 without however being able to justify which of the two. In fact, two of them and two other students initially claimed that it should be 5, without justification because it does not appear in the faces of the given two dice, possibly revealing an influence of the teaching contract in the students' thinking.

> S5: As I can see, only 5 is missing from the numbers. This must be…
> R: Does that matter?

S5: Now that I noticed it, it has sat a little behind my head and it affects me a little. But I will try to see it again.
(S5, T5, F, VA)

Finally, it is necessary to mention as a first approach for solving task 4, a real movement, where a student noted that it is like walking behind the shape and observing its characteristics. As a second approach, student S5 told that he turns the shape 180 degrees and look at the unique cube above and the missing cube at the right part of shape. In his interview, he told that these two approaches applied almost simultaneously but in the end of our conversation, he reported that eventually the first approach preceded.

S5: I try to imagine the shape as if I were going behind it
R: Wait, what do you mean?
S5: Without making a turn, if we thought that this is a three-dimensional object, then it is like standing in front of it and walking to go behind it. Then, I look the position of cube above and how the right part (i.e, the missing cube down) is seemed from the back view.
(S5, T4, S, VA)

5. Conclusions

Students' visual approach and the difficulty of formulating verbal arguments to describe the rotation of a two-dimensional shape, often lead to the use of gestures or body movements to state and justify their argument. On the contrary, in three-dimensional tasks, they have the option to refer to individual characteristics of the shapes and in fact often to combine them with mental rotations following the harmonic type of reasoning as it is defined by Kruktetskii (1976). In fact, the visual strategy seems to precede the students' reasoning and the analytic one to follow. It is worth noting that all students followed a combination of strategies in at least one specific task successfully. If this clue is combined with the fact of problem solving by only three students, where all followed the visual-analytic approach and in fact two of them the creative-constructive solving of task, in a very short time, then we conclude that this combination seems be a strong success predictor of solving a task.

However, the observed inability to hold the transformations of an image subject to mental rotation highlights the correlation of the memory factor with visualization, as Bishop (1983) has pointed out.

At the same time, an important element that emerged from the study is the confusion and difficulty of perception – as previous research points out (Michaelides, 2006) – for some students that the front and back view of a shape are the same and not reversed. However, this element was accompanied in this study by the element of creativity, which is an indication that this misconception may sometimes not be an obstacle to constructive and creative approaches from the beginning. In any case, the role of required action (Gorgorió, 1998) either for interpretation or for construction is important, with construction often requiring interpretation while interpretation not always construction.

Another domain of relative research investigates the geometrical figure apprehension. Geometrical figure apprehension is important for the analysis of a geometrical problem (Duval, 1995). According to Duval (1999), there are four types of cognitive apprehension of a geometrical figure: perceptual, sequential, discursive, and operative. The above theoretical approach of Duval gave rise to dozens of experimental research studies that led to the scientific foundation of the theory (e.g. Gagatsis, Monoyiou, Deliyianni, Elia, Michael, Kalogirou, Panaoura, & Philippou, 2010; Gagatsis, 2011, 2012, 2015; Gagatsis, Michael-Chrysanthou, Deliyianni, Panaoura, & Papagiannis, 2015). These studies, that concern students in primary and secondary and statistical methods like factor analysis and equation structural modeling yield similar results independently of the school level of the students (Michael, Gagatsis, Deliyianni, Monoyiou, & Philippou, 2009; Michael, Gagatsis, Avgerinos, & Kuzniak, 2012; Michael-Chrysanthou, Gagatsis, 2013, 2015). Moreover, in some studies, the geometrical figure apprehension is related to some dimensions of spatial ability (Kalogirou, Elia, & Gagatsis, 2009; Kalogirou & Gagatsis, 2010; 2011). All the above studies on the geometrical figure apprehension and spatial ability concern two-dimensional tasks. The need to move a step forward and expand the above-mentioned studies in three-dimensional tasks arises.

Our article analyses some of the aspects related to the topic of visualization in Mathematics Education, giving a detailed "local" characterization of the study, in reference to some of our research in the field. Of course, the topic "visualization" is extremely vast, and it would be hard to take it completely into account. It would be interesting to frame it also in a broader sense, for instance also in reference to positions that underline that the visual techniques often rely on "not always procedurally 'safe' routines" (Arcavi, 2003). On the other hand, Duval (2005), underlining of the importance for visualization, goes beyond the four types of cognitive apprehension of a geometrical figure and considers the relationship between the dimensions of the representation support (2D) and the objects represented on it (2D or 3D). We believe that it would be important to mention these approaches because our present work is related to them.

Finally, of particular interest in future research is expected to be the focus on the correlation of solution strategies mainly in three-dimensional tasks with the development of students' creativity in terms of flexibility, fluency, and novelty (Torrance, 1974; Silver, 1997; Leikin, 2011). This correlation can reveal the need of differentiated teaching practices, which can encourage the development of spatial reasoning of students. In this direction, differentiation of curriculum so as to include tasks that encourage both visualization and argumentation may help to achieve this goal.

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APPENDIX A

Questionnaire

Task 1
The following figure shows a signal from the traffic code:

ONLY 3 of the following shapes are rotations of the original shape. Find what is NOT!

A B Γ Δ

Task 2
The following figure shows the letter G of the English alphabet:

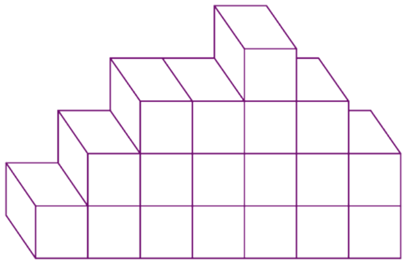
ONLY one of the following is the same as the original shape, but we have just rotated it. Find it!

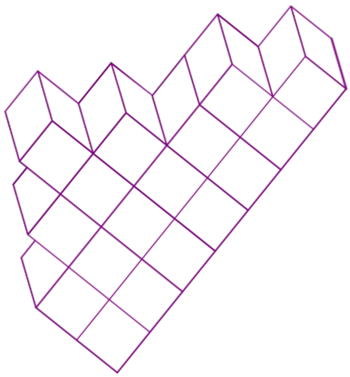
A B Γ Δ

Task 3

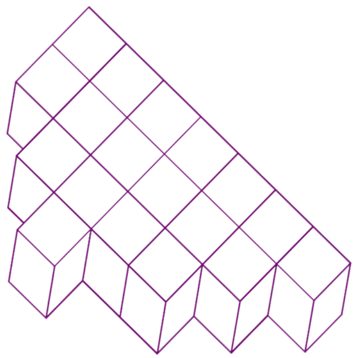
The following shape is given:



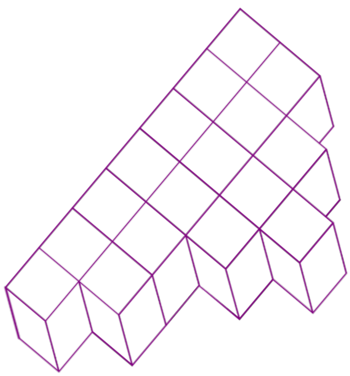
Which of the following shapes is exactly the same as the above shape?



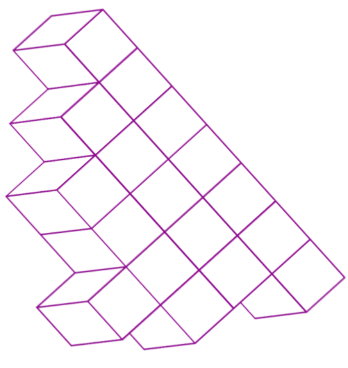
A



B

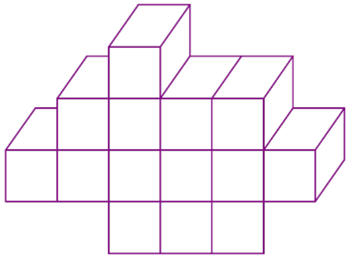


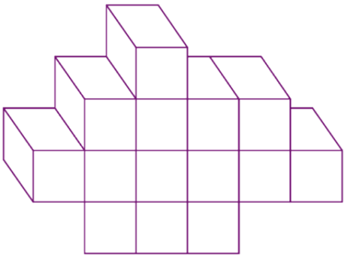
Γ



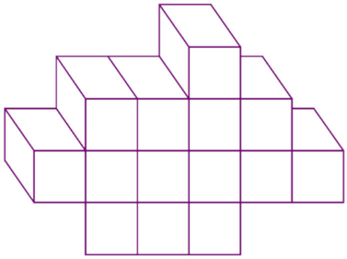
Δ

Task 4
How would the shape seem if we could see it from the back view?

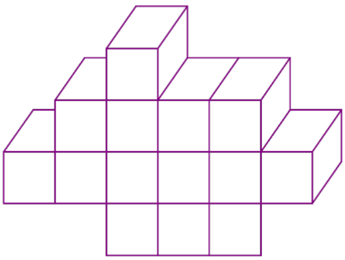




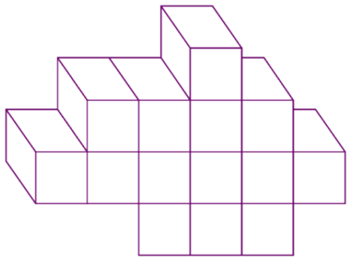
A



B



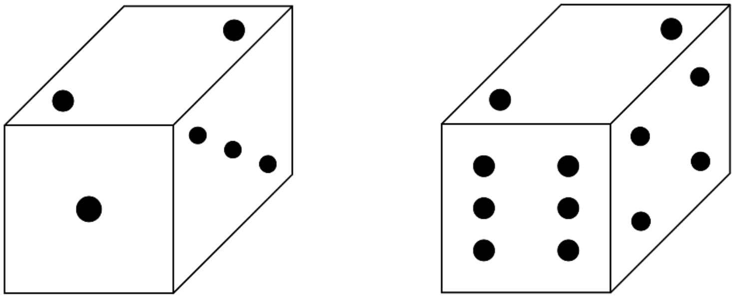
Γ



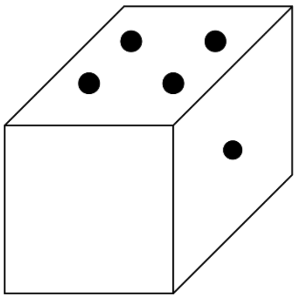
Δ

Task 5

In the figure below you can see the SAME die that have been rolled twice and different faces of it can be seen.



Can you notice the missing face of the die?



Important note

The selection of the above five tasks was based on the analysis protocol of Dr. M. P. Michaelides, Department of Psychology, University of Cyprus. This questionnaire was the result of the master's thesis of Dr. Michaelides at the University of Cambridge under the supervision of Kenneth Ruthven. In this study, the questionnaire was built following that author's protocol with some modifications.

APPENDIX B

Interview questions

Write your answer to the following questions in one letter:

1. Use of Mental Rotation
A. I rotated the whole figure in my mind when making the comparison.
B. I rotated a section of the figure in my mind when making the comparison.
C. I am not sure how I did it.
D. Other.

2. Visual or Analytic strategy or combination of them
A. I thought through the steps verbally in my mind (i.e., "I count the cubes in the second row/column").
B. I relied mainly on visualizing the figures and did not talk myself through the steps.
C. I did both of them.
D. Other.

3. Use of gestures and body movements
A. I used movements of my finger, hand (my body, in general) and/or pencil to help me with the task.
B. I did not use movements of my finger, hand (my body, in general) and/or pencil to help me with the task.

Question	Task 1	Task 2	Task 3	Task 4	Task 5
1					
2					
3					